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NOTES
ON
ROULETTES & GLISSETTES

W. H. BESANT

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ON

ROULETTES AND GLISSETTES.

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PREFACE.

THE following pages contain the explanation of methods, and the investigation of formulæ, which I have for some time past found useful in the discussion of the curves produced by the rolling or sliding of one curve on another.

These methods and formulæ are with a few exceptions original, and, I believe, new; and my object has been to present, from a geometrical point of view, solutions of the various problems connected with Roulettes and Glissettes. I have ventured to introduce, and employ, the word Glissette, as being co-expressive with Roulette, a word which has been in use amongst mathematicians for a considerable time.

The formula of Art. (21) is of course well known; it is given in Salmon's *Higher Plane Curves*, in Walton and Campion's *Solutions*, in Jullien's *Problems*, in Bertrand's *Differential Calculus*, and probably in many other books.

The theorem of Art. (24) was enunciated some years ago, for the particular case of a conic, by Mr Wolstenholme, and extended by myself to the case of any curve. I have however recently found a reference to it in the *Nouvelles Annales* for June, 1869, from which it appears that it was given by Steiner in an early number of the same journal.

For the incisive method of Art. (56) I am indebted to Mr Ferrers.

It will be seen that the general formula of Art. (37) includes most of those which precede it, while it is itself included in that of Art. (55), and that the theorem of Art. (67) reduces all cases of motion in one plane to the cases of Articles (37) or (55).

In a future tract I hope to produce some further developments of the ideas which are here somewhat briefly treated.

W. H. BESANT.

DECEMBER, 1869.

PRELIMINARY REMARKS ON

INFINITESIMALS.

1. AN infinitesimal is a quantity which, under certain assigned conditions, vanishes compared with finite quantities.

If two infinitesimals vanish in a finite ratio to each other, they are said to be of the same order.

Thus, if θ vanish, $\sin\theta$ and θ are of the same order, as are also $\sin m\theta$ and $\tan n\theta$.

If two infinitesimals, α and β, be such that the ultimate ratio of β to α^2 is finite, β is said to be of the second order if α be of the first order.

Thus, $1-\cos\theta$, when θ vanishes, is of the second order if θ be of the first order.

And, generally, an infinitesimal which has, ultimately, a finite ratio to the r^{th} power of another is said to be of the r^{th} order if that other be of the first order.

The order of an infinitesimal is, *à priori*, arbitrary and conventional; but, if any standard be fixed upon, the orders of all others are determinate.

2. Consider figure (1), in which O is the centre of a circle, and AP a small arc; PN, PL perpendiculars on OA and on the tangent at A, and Q the point in which OP produced meets AL.

Then, if $OA = a$, and $AOP = \theta$, it can be shewn by Trigonometry that, when θ is indefinitely diminished,

$$\frac{AL}{AP} = 1, \quad \frac{PL}{AP^2} = \frac{1}{2a}, \quad \frac{QL}{AP^3} = \frac{1}{2a^2},$$

and
$$\frac{PQ - PL}{AP^4} = \frac{1}{4a^3}.$$

Therefore, if AP be an infinitesimal of the first order, AL is of the first order, PL of the second, QL of the third, and $PQ - PL$ of the fourth.

3. If α be an infinitesimal of the first order,

$$\lambda\alpha^2 + \mu\alpha^3 : \nu\alpha^2 :: \lambda : \nu, \text{ ultimately,}$$

$$\lambda\alpha^2 + \mu\alpha^3 \text{ is of the second order:}$$

and generally, it will be seen that the order of an infinitesimal is not affected by the addition to it of an infinitesimal of any higher order.

Let AP' be the arc of a curve, such that AP is its circle of curvature at A; then PP' is of the third order, and therefore, so far as quantities of the 2nd order are concerned, P' may be taken to be coincident with P.

4. If AP, AQ be two infinitesimal arcs, of the first order, of two curves touching each other at A, the distance PQ will be of the second order, and therefore, so far as quantities of the first order are concerned, P and Q may be taken to be coincident.

It will be seen that all the preceding theorems are contained in, or deducible from, the 7th and 11th Lemmas of the first section of the *Principia*.

Thus, from Lemma XI., if AP, AP' be two infinitesimal arcs of the same order, and PL, $P'L'$ the corresponding perpendiculars,

$$PL : P'L' :: AP^2 : AP'^2.$$

ROULETTES.

5. When a curve rolls on a fixed curve any given point in the plane of the rolling curve describes a certain curve, which is called a roulette.

Under the same heading we shall also include the curves enveloped by any given lines, straight or curved, which are carried with the rolling curve.

If a curve roll on a fixed curve, the line joining the point of contact with any point Q in the plane of the rolling curve is the normal to the path of Q.

For, as the curve rolls, the point of the curve, P, in contact with the fixed curve, has no motion, and the whole area is, at the instant, turning round P: hence the direction of motion of Q, i.e. the tangent to its path is at right angles to QP, and QP is the normal. (See fig. 2.)

Thus, if a circle roll on a straight line, and if PD be the diameter through P, $Q'P$ is the normal, and $Q'D$ the tangent to the cycloidal path AQ' of a point Q' of the circumference.

6. *A curve rolls on a straight line; it is required to find the roulette traced by any point* Q.

Let the curve roll from O to P, the point A passing over the point O. (See fig. 2.)

Taking O as the origin, and OP as axis of x, let x, y be co-ordinates of Q.

Then, if $AQP = \theta,\ QP = r,$

$$\tan QPN = \frac{dx}{dy} = r\frac{d\theta}{dr},$$

and $$y = r\sin QPN = r\frac{dx}{ds}.$$

Hence, if the polar equation referred to the point $r = f(\theta)$, be given, we have three equations from which and θ can be eliminated, and the resulting equation will the differential equation to the path of Q.

Or, if the arc $AP\ (=s)$ be found in terms of θ, we ma employ the equations

$$x = s - r\cos NPQ = s - r\frac{dr}{ds},$$

$$y = r\sin QPN = r^2\frac{d\theta}{ds},$$

and the elimination of r and θ will give the equation in x and y, to the path of Q.

If the rolling curve be given by the equation $p = f(r)$, we have

$$p = QN = r\frac{dx}{ds};$$

$$\therefore \text{ since } p = y,$$

$$y = f\left(y\frac{ds}{dx}\right)$$

is the roulette.

7. The two theorems following will be found of great use in the discussion of roulettes.

If (fig. 3) p be the perpendicular from a fixed point (O) on the tangent at any point P of a curve, and ϕ its inclination to a fixed line, then

(1). *The perpendicular from O on the normal* $= \dfrac{dp}{d\phi}$.

(2) *The radius of curvature at* $P = p + \frac{d^2p}{d\phi^2}$.

For, if P' be a consecutive point of the curve, and if the tangents intersect in T, and $YOY' = \delta\phi$,

$$KY' = TK\,\delta\phi,$$

neglecting infinitesimals of the second order.

But $$\delta p = OY' - OY = KY',$$
to the first order;

Therefore, ultimately,

$$dp = PY\,d\phi,$$

or $$OZ = \frac{dp}{d\phi}.$$

Again,

$$\begin{aligned}\delta s = PP' &= PT + TP', \text{ to the first order,}\\ &= KY + TK - PY + P'Y' - TY'\\ &= KY + P'Y' - PY \text{ to the first order,}\\ &= p\,\delta\phi + \delta\,.\,PY\\ &= p\,\delta\phi + \frac{d^2p}{d\phi^2}\,\delta\phi.\end{aligned}$$

Therefore the radius of curvature $= \frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2}$.

If the curve at P be convex to the point O, it will be seen by a figure that the length of the radius of curvature is $-p - \frac{d^2p}{d\phi^2}$.

The sign of $p + \frac{d^2p}{d\phi^2}$ therefore determines whether the curve is concave or convex to the origin.

8. The relation $p = f(\phi)$, connecting p and ϕ, is called the tangential polar equation of the curve*.

---

* This title was suggested by Mr Ferrers. (*Cambridge and Dublin Mathematical Journal*, 1855.)

We may remark that p and ϕ are the polar co-ordinates of the point Y; so that, putting r and θ for p and ϕ, the polar equation of the pedal is

$$r=f(\theta).$$

We also observe that the intrinsic equation is at once obtained by integrating the equation

$$\frac{ds}{d\phi}=p+\frac{d^2p}{d\phi^2},$$

the right-hand member being expressed in terms of ϕ.

9. *To find the tangential polar equation of the roulette traced by a point.*

Let fall OY, OZ, perpendiculars on the tangent QT and the normal QP (fig. 4).

Let $OY=p$, and $YOT=\phi$.

Then

$$\begin{aligned} p &= r - OP\cos\phi, \\ &= r - s\cos\phi, \end{aligned}$$

whence, having r and s in terms of θ, and $\tan\phi$ being $\frac{rd\theta}{dr}$, we can eliminate r and θ and get the relation between p and ϕ.

Or, without finding the arc, we have

$$\begin{aligned} \frac{dp}{d\phi} &= OZ = PZ\tan\phi \\ &= (r-p)\tan\phi, \end{aligned}$$

and, eliminating θ between this equation and

$$\tan\phi=r\frac{d\theta}{dr},$$

we get the differential equation, in p and ϕ, of the roulette.

10. Ex. 1. *To find the roulette traced by the focus of a parabola rolling on a straight line.*

In this case

$$\frac{2a}{r} = 1 + \cos\theta,$$

the point A, fig. (2), being the vertex of the curve, and Q the focus;

$$\therefore\ \tan QPN = \cot\frac{\theta}{2} = \frac{dx}{dy},$$

and
$$y = r\sin QPN = \frac{a}{\cos\frac{\theta}{2}};$$

$$\therefore\ \left(\frac{dx}{dy}\right)^2 = \frac{a^2}{y^2 - a^2},$$

whence, by integration,

$$y = \frac{a}{2}\left(\epsilon^{\frac{x}{a}} + \epsilon^{-\frac{x}{a}}\right).$$

That is, the roulette is a catenary.

Ex. 2. *The curve $r = a$ versin θ rolls on its axis; required the locus of its pole.*

Taking the second system of equations of Art. (6), we find that

$$QPY = \frac{\theta}{2},$$

and
$$s = 4a\left(1 - \cos\frac{\theta}{2}\right);$$

$$\therefore\ x = 4a\left(1 - \cos\frac{\theta}{2}\right) - a\,(1 - \cos\theta)\cos\frac{\theta}{2}$$

$$= 4a - 2a\cos\frac{\theta}{2}\left(2 + \sin^2\frac{\theta}{2}\right),$$

and $$y = 2a\sin^3\frac{\theta}{2};$$

whence $$4a - x = \sqrt{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}}\,\{2\,(2a)^{\frac{2}{3}} + y^{\frac{2}{3}}\}.$$

11. If the roulette be given in terms of x and y, we can at once find the rolling curve.

For, Art. (6),

$$p = y, \text{ and } p = r\frac{dx}{ds}.$$

Hence, if $y = f(x)$ be the roulette, we can eliminate x and y, and find the equation, in p and r, to the rolling curve, referred to the carried point as origin.

Ex. 1. If the roulette be the catenary

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right),$$

$$\frac{dx}{ds} = \frac{c}{y}, \text{ and } p = r.\frac{c}{y} = \frac{rc}{p},$$

$$\therefore\ p^2 = rc,$$

that is, the rolling curve is a parabola.

Ex. 2. If the roulette be

$$y^2 = 4ax,$$

$$\left(\frac{r}{p}\right)^2 = \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4a^2}{y^2};$$

$$\therefore\ p^2 = r^2 - 4a^2,$$

the involute of a circle.

Ex. 3. If the roulette be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the curve is

$$r^2 = e^2p^2 + \frac{b^4}{a^2},$$

an epicycloid; (Art. 14).

Ex. 4. If the roulette be

$$\frac{ds}{dx}=f(y),$$

the curve is

$$r=pf(p).$$

12. The equations in x and y, or in polar co-ordinates, to the roulettes produced by circles rolling on straight lines or on circles are at once obtained from figures.

Thus, for an epicycloid, if a and b be the radii of the fixed and rolling circles, (see fig. 5),

$$AOP=\theta,\ QCP=\frac{a\theta}{b};$$

$$\therefore\ x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta,$$

$$y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta.$$

The area swept over by the radius vector is most easily found by help of these equations and the expression

$$\frac{1}{2}\int(xdy-ydx).$$

13. The roulettes traced by the centres of circles rolling on curves belong to the class of 'parallel curves.'

If b be the radius of the circle, x, y the co-ordinates of its centre, and x', y' of the point of contact,

$$x=x'+b\cos\phi,$$
$$y=y'+b\sin\phi,$$

where ϕ is the inclination of the normal to the axis of x.

Or, if the given curve be

$$p=f(\phi),$$

the parallel is

$$p=f(\phi)+b.$$

Thus the parallels of a parabola and of an ellipse are

$$p = a \sec \phi + b,$$

$$p = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} + d.$$

14. *To find the tangential polar equation, and the intrinsic equation, of an epicycloid.*

In (fig. 5), let $OA = a$, $CP = b$, and $ACP = \theta$. Then, if Q be the point tracing out the epicycloid, starting from A, QD is the tangent at Q, and if $p = OY$, the perpendicular on QD, and $AOY = \frac{\pi}{2} - \phi$, so that ϕ is the angle of deflection from A,

$$\phi - \theta = CDQ = \frac{a\theta}{2b}, \text{ and } p = (a + 2b) \sin \frac{a\theta}{2b};$$

$$\therefore\ p = (a + 2b) \sin \frac{a\phi}{a + 2b}.$$

Hence $$\frac{ds}{d\phi} = (a + 2b) \left\{1 - \frac{a^2}{(a + 2b)^2}\right\} \sin \frac{a\phi}{a + 2b}$$

$$= \frac{4b(a + b)}{a + 2b} \sin \frac{a\phi}{a + 2b},$$

and $$s = \frac{4b}{a}(a + b)\left(1 - \cos \frac{a\phi}{a + 2b}\right),$$

measuring the arc from the point A*.

We may observe that the radius of curvature of the epicycloid at Q

$$= \frac{ds}{d\phi} = \frac{4b(a + b)}{a + 2b} \sin PDQ,$$

$$= \frac{2(a + b)}{a + 2b} PQ.$$

---

* Hence $s : 2 . \text{vers} . \text{arc } PQ :: 2 . OC : OP$, the form given in the Principia, Section X, Prop. XLVIII.

To find the equation, in p *and* r, *of an epicycloid.*

If $OQ=r$, we have

$$\begin{aligned} r^2 &= (a+b)^2+b^2-2b(a+b)\cos\frac{a\theta}{b} \\ &= a^2+4b(a+b)\sin^2\frac{a\theta}{2b} \\ &= a^2+4b(a+b)\frac{p^2}{(a+2b)^2}, \end{aligned}$$

or
$$p^2=\frac{(a+2b)^2}{4b(a+b)}(r^2-a^2).$$

15. If a circle of radius b roll inside, or more generally with its concavity in the same direction, on a circle of radius a, the hypocycloid generated by a point in the circumference will be

$$\frac{ds}{d\phi}=4b\frac{a-b}{a-2b}\sin\frac{a\phi}{a-2b}.^*$$

If $b>a$, this may be written in the form

$$\frac{ds}{d\phi}=-4(b-a)\frac{a+(b-a)}{a+2(b-a)}\sin\frac{a\phi}{a+2(b-a)},$$

so that, when $b>a$, the hypocycloid is identical with the epicycloid generated by a circle of radius $b-a$ rolling outside a circle of radius a.

This can also be seen by direct geometry; for if PQ meet the fixed circle in R (fig. 6), let OR produced meet DQ produced in E; then RE is the diameter of a circle, touching at R and passing through Q.

The angle $FQR=FRQ=RPO$; therefore FQ is parallel to OP; and

$$\angle REQ=\frac{\pi}{2}-ERQ=PDQ;$$

---

* The p and r equation is $p^2=\frac{(a-2b)^2}{4b(a-b)}(a^2-r^2)$, and, if $2b=a-c$, this becomes $p^2=\frac{c^2}{a^2-c^2}(a^2-r^2)$, the form given by the Jesuit Fathers in the notes to Prop. LI. of the Principia.

$$\therefore\ RE = OE - OR = 2(b-a),$$

and $OF = b = CQ$, so that OF is parallel to CQ.

Hence $$\text{arc } RQ = (b-a)\,.\,POR = (b-a)\,PCQ$$
$$= (b-a)\,.\,\frac{a\theta}{b} = \text{arc } RA,$$

so that the point Q, carried by the circle F, will produce the hypocycloid.

16. It may be useful to give the several equations for the case of a three-cusped hypocycloid, a curve possessing many remarkable properties.

If $AOP = \theta$, and ϕ be the angle of deflection from OA of the tangent EQ, (fig. 7),

$$QCP = 3\theta, \text{ and } \phi = \frac{3\theta}{2} - \theta = \frac{\theta}{2},$$

$$p = OY = OP\sin CPQ = a\cos\frac{3\theta}{2} = a\cos 3\phi,$$

and $$\frac{ds}{d\phi} = -8a\cos 3\phi;$$

$$\therefore\ s = \frac{8a}{3}(1 - \sin 3\phi) = 8b(1 - \sin 3\phi).$$

Again, $$r^2 = 4b^2 + b^2 + 4b^2\cos 3\theta$$
$$= b^2 + 8b^2\frac{p^2}{9b^2} = b^2 + \frac{8}{9}p^2,$$

or $$8p^2 = 9(b^2 - r^2).$$

17. *A curve rolls on a given curve, it is required to find the path of any point in its plane.*

If $O'P$ be the rolling curve, O' having been coincident with O (fig. 8), let x, y be coordinates of P referred to the normal and tangent at O, and x', y' referred similarly to O'; ϕ, ϕ' the angles which the normal at P makes with the normals at O, O'.

Then, if ξ, η be co-ordinates ON, $O'N$ of O',

$$\xi = y' \sin(\phi + \phi') - x' \cos(\phi + \phi') - x,$$
$$\eta = y - y' \cos(\phi + \phi') - x' \sin(\phi + \phi'),$$

and, as the right-hand members of these equations can be found in terms of the arc $OP\,(s)$, the relation between ξ and η can be found.

If Q be a point, the co-ordinates of which, referred to O', are (a, b), and if ξ', η' be co-ordinates of Q referred to O,

$$\xi' = \xi + a \cos(\phi + \phi') - b \sin(\phi + \phi'),$$
$$\eta' = \eta + a \sin(\phi + \phi') + b \cos(\phi + \phi').$$

18. *A curve rolls on a straight line; it is required to find the curvature of the path of any point carried with it.*

Let QP, $Q'P'$ be consecutive normals to the path of Q (fig. 9).

Their intersection E is ultimately the centre of curvature.

Let Qp be the changed position of QP.

Then $Q'LQ =$ the angle through which the curve has turned

$=$ the angle through which the normal at P has turned

$= \dfrac{\delta s}{\rho}$, if ρ be the radius of curvature at P;

$$\therefore\ EQ = \frac{\text{arc } QQ'}{\angle E} = \frac{QQ'}{QLQ' - LQP}$$

$$= \frac{QL \,.\, \dfrac{\delta s}{\rho}}{\dfrac{\delta s}{\rho} - \dfrac{\delta s \,.\, \cos\alpha}{r}},$$

if α be the angle between $QP\,(r)$ and the normal at P,

$$= \frac{r^2}{r - \rho\cos\alpha},$$

ultimately, observing that since the displacement of p from P' is of the second order, we may in this case assume that p and P' are coincident; Arts. (3) and (4).

Thus, in the case of a cycloid,

$$r = 2a \cos \alpha, \text{ and } \rho = a;$$

$$\therefore \; EQ = 2r = 2PQ.$$

19. The preceding formula may also be obtained by help of the equation

$$\frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2}.$$

For, (fig. 10), if $OPQ = \phi$, and $OP = AP = s$,

$$p = r - s \cos \phi,$$

and $$\frac{dp}{d\phi} = OZ = s \sin \phi;$$

$$\therefore \; p + \frac{d^2p}{d\phi^2} = r + \sin \phi \frac{ds}{d\phi}.$$

But if $AQP = \theta$, and $AFP = \psi$,

$$\theta = \psi + \frac{\pi}{2} - \phi;$$

therefore radius of curvature of roulette

$$= r + \cos \alpha \frac{ds}{d\psi - d\theta};$$

and since $r \dfrac{d\theta}{ds} = \sin \phi$, and $\dfrac{ds}{d\psi} = \rho$,

this $$= r + \frac{\cos \alpha}{\dfrac{1}{\rho} - \dfrac{1}{r} \cos \alpha} = \frac{r^2}{r - \rho \cos \alpha}.$$

20. *If a curve roll on a fixed curve over a small arc* δs, *the angle turned through by any line in the plane of the rolling curve is* $\delta s \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)$, *where* ρ *and* ρ' *are the radii of curvature at the point of contact.*

Let $Pp = PP' = \delta s$, (fig. 11), and let the normals Op, $O'P'$ meet in L.

Then the angle between these, OLK, is the angle required, and this angle

$$= POp + P'O'P,$$

$$= \frac{\delta s}{\rho} + \frac{\delta s}{\rho'}.$$

21. *A curve rolls on a fixed curve; to find the curvature of the roulette traced by any point carried with it.*

The angle QLQ' = the angle of displacement

$$= \delta s \left(\frac{1}{\rho} + \frac{1}{\rho'}\right), \text{ (fig. 12)},$$

and $$EQ = \frac{QQ}{QEQ'} = \frac{QL\,(QLQ')}{QLQ' - LQP},$$

and, since the displacement of p is of the second order,

$$LQP' = \frac{PP' \cos\alpha}{P'Q'} = \frac{\delta s \,.\, \cos\alpha}{r};$$

$$\therefore\; R = EQ = \frac{r\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)}{\frac{1}{\rho} + \frac{1}{\rho'} - \frac{\cos\alpha}{r}}.$$

22. Referring to the same figure, let ψ be the inclination of QP to the fixed normal OF, A the point which has passed over O, $AQP = \theta$, $QP = r$.

Then, if $r = f(\theta)$ be known, and $s = F(\phi)$ be the equation to OP, EQ can be found in terms of s and therefore of ϕ.

Also $$\psi = \phi - \alpha;$$

therefore $\frac{d\sigma}{d\psi}$ or EQ can be found in terms of ψ, and this gives the intrinsic equation to the path of Q.

23. Thus, for an epicycloid,

radius of curvature of path of Q, (fig. 5),

$$=PQ\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}-\frac{\cos\alpha}{PQ}},$$

or
$$R=PQ.\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}+\frac{1}{2b}}=2PQ.\frac{a+b}{a+2b}.$$

Again, take the case of an ellipse rolling on an equal ellipse, corresponding points being in contact, and consider the path of a focus S.

If P be the point of contact,

$$\cos\alpha=\frac{PF}{AC},\quad \rho=\rho'=\frac{CD^2}{PF};$$

$$\therefore\ R=\frac{2\frac{PF}{CD^2}.SP}{\frac{2PF}{CD^2}-\frac{PF}{AC.SP}}=2AC,$$

as is *à priori* obvious.

24. *If a curve roll on a straight line, the arc of the roulette is equal to the corresponding arc of the pedal.*

The angle turned through, (fig. 13),

$$\begin{aligned}&=\text{angle between normals at } P \text{ and } P'\\ &=YQY';\end{aligned}$$

$$\begin{aligned}\therefore\ \text{arc } QQ' &= QP.\,YQY'\\ &= QT.\,YQY'\\ &=\frac{YY'}{\sin YQY'}.\,YQY'=YY';\end{aligned}$$

QT being the diameter of the circle about YQY'.

Or thus, if $QY = y$,

$$\frac{dy}{ds} = \cos QPY,$$

and if σ = arc of pedal, y being the radius vector,

$$\frac{dy}{d\sigma} = \cos QPY;$$

$$\therefore\ ds = d\sigma.$$

25. *If a curve roll on a fixed curve, the element of arc of the roulette is to the corresponding arc of the pedal as* $\rho + \rho' : \rho'$, ρ *being the radius of curvature of the rolling curve, and* ρ' *of the fixed curve.*

Imagining the line OT to be the fixed curve (fig. 13), the angle turned through $= ds\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)$;

$$\therefore\ QQ' = QP.ds\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Also $$YY' = QP.\sin YQY' = QP.\frac{ds}{\rho};$$

$$\therefore\ QQ' : YY' = \frac{1}{\rho} + \frac{1}{\rho'} : \frac{1}{\rho}$$
$$= \rho + \rho' : \rho'.$$

Hence the length of the arc of the roulette

$$= \int \left(1 + \frac{\rho}{\rho'}\right) d\sigma,$$

where $d\sigma$ is an element of the arc of the pedal.

In the case of cycloidal, or trochoidal curves, ρ and ρ' are constant, and the arc of the roulette is proportional to the arc of the pedal.

In the case of a curve rolling on an equal curve, corresponding points being in contact, the arc of the roulette is always double that of the pedal.

26. *If a curve roll on a straight line, the area between th roulette, the fixed line, and any two ordinates, is double the cor responding pedal area.*

For, if $Y'N$ be the perpendicular from Y' on PY (fig 14), the element of area $= QY \,.\, YN$, neglecting infinitesimals of the second order,

$$\begin{aligned} &= QY \,.\, YY' \,.\cos TYY' \\ &= QY \,.\, YY' \,.\sin QYY' \\ &= 2\Delta QYY'. \end{aligned}$$

Or thus, if x, y be co-ordinates of Q,

$$\frac{ydx}{ds} = y \sin QPY = y \sin QYZ = QZ = p;$$

$$\begin{aligned} \therefore\ ydx &= pds = pd\sigma \\ &= 2 \text{ (element of polar area).} \end{aligned}$$

27. *To find the area swept over by the normal* QP.

Taking figure (14), let $QP = r$, $PQp = \delta\theta$, and $\delta\phi =$ the angle of deflection from P to p, which is equal to the angle between Qp and $Q'P'$.

Then the area $QPP\,Q' = QPP' + QP'Q'$

$$= QPp + QpQ',$$

observing that pP' is of the second order,

$$= \frac{1}{2} r^2\delta\theta + \frac{1}{2} r^2\delta\phi;$$

therefore the area between the roulette, the fixed line and two normals

$$= \frac{1}{2}\int r^2 d\theta + \frac{1}{2}\int r^2 d\phi.$$

If the curve be a closed curve and make one revolution,

this area = area of curve $+ \frac{1}{2}\int_0^{2\pi} r^2 d\phi$.

Hence, if the rolling commence when QP is perpendicular to the fixed line,

$$2\,(\text{area of pedal}) = \text{area of curve} + \frac{1}{2}\int_0^{2\pi} r^2 d\phi\,;$$

$$\text{or area of curve} = \int_0^{2\pi}\left(p^2 - \frac{1}{2}r^2\right)d\phi.$$

28. *A curve rolls on a fixed curve; to find the area swept over by the normal* QP.

If the arc $Pp = PP'$, fig. (15), then, pP' being of the second order,

$$\begin{aligned}\text{area } QPP'Q' &= QPp + QpQ'\\ &= \frac{1}{2}r^2\delta\theta + \frac{1}{2}r^2\left(\frac{\delta s}{\rho} + \frac{\delta s}{\rho'}\right);\end{aligned}$$

therefore area swept over by QP

$$= \int \frac{1}{2}r^2 d\theta + \int \frac{1}{2}r^2 ds\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Hence, if the curve be a closed oval, and if it make a complete revolution, the area between the arc of the roulette, the two normals at its ends, and the curve

$$= \text{area of curve} + \frac{1}{2}\int r^2\delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

29. *To find the locus of the centre of curvature at the point of contact of a curve rolling on a straight line.*

Let x, y be co-ordinates of the centre of curvature, then, if $s = f(\phi)$ be the rolling curve,

$$x = s = f(\phi),$$

and

$$y = \rho = f'(\phi),$$

whence, eliminating ρ, the locus is obtained.

Thus, if the curve be an epicycloid, or hypocycloid, the locus is an ellipse.

If it be a catenary, the locus is a parabola, and if it be an equiangular spiral, the locus is a straight line.

30. If the curve roll on a fixed curve, $s=F(\phi')$, and if $s=f(\phi)$ be the rolling curve,

$$x=OM+\rho\sin\phi' \text{ (fig. 16)},$$

$$y=\rho\cos\phi'-PM;$$

therefore, if OM and PM can be found in terms of ϕ', we have, with $f(\phi)=F(\phi')$, three equations from which ϕ and ϕ' may be eliminated.

Suppose, for example, the curves to be equal catenaries, their vertices at first coinciding.

Then, $\rho=c\sec^2\phi$, and $PN=c\sec\phi-c$;

$$\therefore\; y=c.$$

The locus is therefore a straight line, as is *à priori* obvious, if we remember that the normal bounded by the directrix is equal to the radius of curvature.

31. To find the length of the curve formed by the successive positions of the centre of curvature we may proceed as follows.

Let Q be the centre of curvature at P, q at p, and Q' the position of q when the curve has rolled from P to P', so that QQ' is an element of the locus (fig. 17).

Then q may be taken to be in the normal PQ, since its distance from PQ, the tangent to the evolute at q, is an infinitesimal of the second order.

Hence, if $PQ=\rho$, and if ρ' be the radius of curvature of the fixed curve at P,

$$Qq=\delta\rho,\;\; qQ'=(\rho+\rho')\frac{ds}{\rho'},$$

and the inclination to PQ of the tangent at Q

$$=\tan^{-1}\frac{(\rho+\rho')\,ds}{\rho' d\rho}.$$

Also, if $$QQ' = \delta\sigma,$$

$$\delta\sigma^2 = \delta\rho^2 + \left(\frac{\rho+\rho'}{\rho'}\right)^2 \delta s^2.$$

Hence we can find the intrinsic equation, for if ψ be the inclination of the tangent QQ' to the tangent at a fixed point O,

$$\psi = \phi - \frac{\pi}{2} + \tan^{-1}\frac{(\rho+\rho')\,ds}{\rho'\,d\rho},$$

ϕ being the deflection of the fixed curve from O to P.

As an example, again take the case of the equal catenaries; then

$$\rho = \rho' = c\sec^2\phi, \quad \delta s = \rho\delta\phi,$$

and $$\psi = \phi - \frac{\pi}{2} + \tan^{-1}(\cot\phi) = 0,$$

so that QQ' is parallel to the tangent at O, as already seen.

32. We have hitherto considered only the roulettes produced by points carried with a rolling curve; we now proceed to consider the roulettes enveloped by lines carried with a rolling curve.

A curve rolls on a straight line, to find the envelope of any straight line carried with it.

If P be the point of contact, and PQ the perpendicular let fall from P on the carried line, Q is the point of contact of its envelope (fig. 18).

Let pq be the perpendicular from a consecutive point p, then as the curve rolls over PP', q is carried to Q', and if $\delta\sigma$ be an arc of the roulette enveloped,

$$\begin{aligned}\delta\sigma &= Qq + qQ' \\ &= \sin\phi\delta s + r\delta\phi,\end{aligned}$$

if $OPQ = \phi$, and $PQ = r$.

Hence the radius of curvature $= \dfrac{d\sigma}{d\phi}$

$$= r + \sin\phi\,\frac{ds}{d\phi}$$

$$= r + \rho\sin\phi.$$

For example, consider the roulette produced by a diameter of a circle rolling on a straight line.

Then $$r = a\sin\phi,\quad \rho = a,$$

$$\frac{d\sigma}{d\phi} = 2a\sin\phi,$$

and the roulette is a cycloid.

Ex. 2. *A parabola rolls on a straight line, it is required to find the envelope of the latus rectum.*

In this case, $\rho\sin\phi = 2SP$ (fig. 19), and

$$PQ = a - x;$$

$$\therefore\ \frac{d\sigma}{d\phi} = a - x + 2\,(a + x)$$

$$= 3a + a\cot^2\phi,$$

$$\therefore\ = 2a + a\,\mathrm{cosec}^2\phi;$$

$$\therefore\ \sigma = 2a\phi - a\cot\phi + C,$$

and the length of the roulette between the two points at which it cuts the fixed line, i.e. from

$$\phi = \frac{\pi}{4} \text{ to } \phi = \frac{3\pi}{4}, \text{ is } (\pi + 2)\,a.$$

33. The tangential polar equation may be obtained, directly.

Thus, if $s = OP =$ arc AP, and $p = OY$ the perpendicular on the carried line,

$$p = r - s\cos\phi,$$

whence the equation, if r and s be known in terms of ϕ.

Or, if s only be known, and OZ be the perpendicular on PQ,

$$OZ = \frac{dp}{d\phi},$$

and therefore $$\frac{dp}{d\phi} = s \sin \phi.$$

Hence the radius of curvature

$$\begin{aligned} &= p + \frac{d^2p}{d\phi^2} \\ &= r - s\cos\phi + s\cos\phi + \sin\phi\,\frac{ds}{d\phi} \\ &= r + \rho\sin\phi, \end{aligned}$$

as before.

34. *A curve rolls on a given curve, carrying a straight line; to find the roulette enveloped.*

Making the same construction as before, and observing that the displacement of p is of the second order (fig. 20),

$$\begin{aligned} \delta\sigma &= Qq + qQ' \\ &= \delta s \cos\alpha + r\delta\phi, \end{aligned}$$

where $\delta\phi = \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)$, and α is the inclination of PQ to the normal at P.

Hence the radius of curvature of the roulette at Q

$$= \frac{d\sigma}{d\phi} = r + \cos\alpha\,\frac{\rho\rho'}{\rho+\rho'},$$

and if r, α, ρ and ρ' can be found in terms of the angle which PQ makes with some fixed line, the intrinsic equation can be found.

35. EXAMPLE. *A circle rolls outside a fixed circle; to find the length of the curve enveloped by a diameter.*

If AD be the diameter, A passing over A', and if

$A'OP=\theta$ (fig. 21),

$$PQ=b\sin\frac{a\theta}{b},$$

and
$$\frac{d\sigma}{d\phi}=b\sin\frac{a\theta}{b}+\cos CPQ\,\frac{ab}{a+b}$$
$$=\frac{b^2+2ab}{a+b}\sin\frac{a\theta}{b}.$$

The angle of deflection of AD from OA'

$$=\phi=\theta+\frac{a\theta}{b};$$

$$\therefore\ \frac{d\sigma}{d\phi}=\frac{b^2+2ab}{a+b}\sin\frac{a\phi}{a+b}\ldots\ldots\ldots\ldots\ldots(\alpha),$$

and
$$\sigma=\frac{b^2+2ab}{a}\left(1-\cos\frac{a\phi}{a+b}\right),$$

measuring from A'.

Taking a half roll of the circle, that is from $\theta=0$ to $\theta=\frac{\pi b}{a}$, we get the length of the arc from one cusp to the next, which is therefore

$$2\frac{b}{a}(b+2a).$$

Comparing the equation (2) with that of Art. (14), we observe that the envelope of AD is the epicycloid which would be produced by a circle of radius $\frac{b}{2}$ rolling on the circle O.

This can also be seen geometrically, for, describing a circle on PC as diameter, the arc PQ is equal to the arc PA, and therefore to the arc PA'.

36. *A curve rolls on an equal curve, corresponding points coinciding; to find the envelope of any normal of the rolling curve.*

Let $\phi = BTQ$ (fig. 22), and measure ψ from the tangent at O, the point corresponding to A.

Then, for the envelope of the normal at A,

$$\frac{d\sigma}{d\phi} = r + \cos\alpha\,\frac{\rho}{2}, \quad \text{Art. (34)},$$

and, if $s = f(\psi)$ be the curve,

$$r = \int_0^\psi \cos\psi\, ds, \quad \rho = f'(\psi), \quad \text{and} \quad \phi = 2\psi;$$

$$\therefore \frac{d\sigma}{d\phi} = \int_0^{\frac{\phi}{2}} \cos\psi f'(\psi)\, d\psi + \frac{1}{2}\sin\frac{\phi}{2} f'\left(\frac{\phi}{2}\right).$$

For example, let a circle roll on an equal circle; then

$$s = a\frac{\phi}{2},$$

$$\frac{d\sigma}{d\phi} = \int_0^{\frac{\phi}{2}} a\cos\theta\, d\theta + \frac{a}{2}\sin\frac{\phi}{2} = \frac{3a}{2}\sin\frac{\phi}{2},$$

and $$\sigma = 3a\left(1 - \cos\frac{\phi}{2}\right),$$

a two-cusped epicycloid.

Taking a half roll of the circle the arc is $6a$, which agrees with the result in Art. (35), putting $b = a$.

37. *A curve rolls on a fixed curve; it is required to find the envelope of any curve carried with it.*

If P be the point of contact, draw from P normals to the carried curve (fig. 23).

Then, if PQ be one of these normals, it is the normal at Q to the envelope, and the other normals similarly belong to other portions of the envelope.

Let pq, the normal from a consecutive point p, roll into the position $P'Q'$; then E, the intersection of PQ and $P'Q'$, is the centre of curvature.

The arc $QQ' = \delta\sigma = Qq + r\delta\phi$, where $r = PQ$ and

$$\delta\phi = \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Let ρ'' be the radius of curvature at Q of the carried curve; then

$$Qq : \delta s . \cos\alpha :: \rho'' : r + \rho'',$$

α being the inclination of QP to the normal at P;

and the angle $PEP' = \delta\phi - \dfrac{Qq}{\rho''}$

$$= \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) - \delta s\,\frac{\cos\alpha}{r + \rho''};$$

therefore EQ, the radius of curvature of the envelope,

$$= \frac{\dfrac{\rho''\cos\alpha}{r + \rho''} + r\left(\dfrac{1}{\rho} + \dfrac{1}{\rho'}\right)}{\dfrac{1}{\rho} + \dfrac{1}{\rho'} - \dfrac{\cos\alpha}{r+\rho''}}.$$

Making ρ'' infinite, this of course gives the formula of Art. (34), and if ρ'' vanish, the formula of Art. (21) results.

If the various quantities involved in this expression can be found in terms of ψ, the angle of deflection of PQ, the intrinsic equation is determined.

If any of the curves instead of being convex, as in the figure, be concave, the signs of ρ, &c. must be changed.

38. Ex. 1. *A curve rolls on another, carrying with it a parallel curve.*

In this case $\alpha = 0$, $r = d$, and $\rho'' = \rho - d$.

Hence EQ becomes $d + \rho'$, a result which is obviously true.

Ex. 2. *A straight line rolls on a fixed circle, carrying an equal circle with which it is in contact.*

Let $A'OP = \theta$ (fig. 24), then

$$r = CP - a = a\sqrt{1+\theta^2} - a,$$

$$\cos a = \frac{a}{CP} = \frac{1}{\sqrt{1+\theta^2}},$$

$$\rho = \infty,\ \rho' = \rho'' = a,$$

and therefore $$\frac{d\sigma}{d\psi} = a\,\frac{\dfrac{1}{1+\theta^2} + \sqrt{1+\theta^2} - 1}{1 - \dfrac{1}{1+\theta^2}}$$

$$= a\left(\frac{\overline{1+\theta^2}\,\big|^{\frac{3}{2}} - \theta^2}{\theta^2}\right).$$

Also, if ψ be the inclination of PQ to OA',

$$\psi = \theta - \alpha = \theta - \tan^{-1}\theta;$$

$$\therefore d\psi = d\theta - \frac{d\theta}{1+\theta^2},$$

and $$\frac{d\sigma}{d\theta} = a\left(\frac{\overline{1+\theta^2}\,\big|^{\frac{3}{2}}}{\theta^2} - 1\right)\frac{\theta^2}{1+\theta^2} = a\left(\sqrt{1+\theta^2} - \frac{\theta^2}{1+\theta^2}\right).$$

This equation, when integrated, determines the length of the envelope.

If we put $-a$ for ρ'', we shall obtain the other portion of the envelope due to the normal PQ'.

39. *A curve rolls on a straight line; to find the area between the straight line, the envelope of any carried straight line, and two normals of the envelope.*

The element of area $PQQ'P'$ (fig. 18),

$$= PQqp + P'qQ'$$

$$= r\delta s \sin\phi + \frac{1}{2}r^2\delta\phi$$

$$= \left(r\rho\sin\phi + \frac{1}{2}r^2\right)\delta\phi,$$

the integration of which expression gives the area if the intrinsic equation if the rolling curve be known.

If the line PQ fall below the line OP, the element of area swept over will be

$$\left(r\rho\sin\phi - \frac{1}{2}r^2\right)\delta\phi.$$

This however is included in the former, if we suppose r to be an algebraic expression for PQ.

40. *A curve rolls on another; to find the area between the envelope of any carried curve, any two normals of the envelope, and the fixed curve.*

The element of area $= PQQ'P'$ (fig. 23),

$$= PQqp + \frac{1}{2}r^2\,\delta\phi,$$

where $$\delta\phi = \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Hence the area swept over exceeds the area between the curve and the carried curve by $\int \frac{1}{2}r^2\,d\phi$.

Thus in the case of Ex. 2 of Art. (38),

$$r^2 = a^2\left(2 + \theta^2 + 2\sqrt{1+\theta^2}\right),$$

and $$d\psi = d\theta;$$

therefore the area of the envelope exceeds the area APQ by

$$\frac{a^2}{2}\int\left(2 + \theta^2 + 2\sqrt{1+\theta^2}\right)d\theta.$$

41. Adopting the notation of preceding articles, we can give an expression for the element $PQqp$ (fig. 23).

For this element

$$= \left(\frac{1}{2}(r+\rho'')^2 - \frac{1}{2}\rho''^2\right)\frac{\delta s''}{\rho''}$$

$$= \left(r\rho'' + \frac{r^2}{2}\right)\frac{\delta s''}{\rho''},$$

and $\qquad \delta s'' : \delta s \,.\, \cos\alpha :: \rho'' : r+\rho'';$

therefore element $PQqp = \left(\rho'' + \frac{r}{2}\right)\frac{r\cos\alpha\delta s}{r+\rho''}$.

42. The following examples will serve as additional illustrations of the preceding methods.

A cycloid rolls on a straight line; it is proposed to consider the roulette enveloped by the tangent at its vertex.

The cycloid is $s = 4a\sin\phi$, and for the envelope of AQ (fig. 25),

$$\frac{d\sigma}{d\phi} = \rho\cos\phi - PQ$$

$$= 4a\cos^2\phi - 2a\sin^2\phi$$

$$= a + 3a\cos 2\phi;$$

$$\therefore\ \sigma = a\phi + \frac{3a}{2}\sin 2\phi,$$

measuring from O.

To trace the curve, observe that there is a cusp when

$$\cos 2\phi = -\frac{1}{3},$$

i.e., when ϕ is a little greater than $\frac{\pi}{3}$, and that the curve cuts the initial line at distances $2a$ from O: also that for one roll of the cycloid the curve lies wholly below the initial line.

If the cycloid be continued, and the rolling go on continuously, the next branch of the cycloid will give the next half of the roulette above the initial line, after which the successive branches of the cycloid will produce continually the same roulette.

Further, the evolute of the roulette is

$$s = 3a \cos 2\phi,$$

a four-cusped hypocycloid.

The curve may be further examined by finding x and y, the co-ordinates of Q referred to O, viz.

$$x = 4a \sin\phi - 2a \sin^3\phi, \qquad y = a \sin\phi \sin 2\phi.$$

Fig. (26) represents the roulette, and its evolute.

The element of area swept over by PQ

$$= PQ\delta s \cos\phi - \frac{1}{2} PQ^2 \delta\phi$$

$$= 4a^2 \sin^2\phi \,(2 - 3\sin^2\phi)\, \delta\phi,$$

which becomes negative when $\sin^2\phi > \frac{2}{3}$, or when

$$\cos 2\phi = -\frac{1}{3},$$

i. e. at the cusp.

The integration from $\phi = 0$ to $\sin^2\phi = \frac{2}{3}$ gives the area OET, and from $\sin^2\phi = \frac{2}{3}$ to $\phi = \frac{\pi}{2}$ the difference between the areas AET and CET.

If we wish to find the area enclosed by the roulette, it will be at once given by integrating the expression,

$$\frac{1}{2}(xdy - ydx), \text{ from } \phi = 0 \text{ to } \phi = \frac{\pi}{2}.$$

This expression

$$= a^2 (6 \sin^4\phi - 4 \sin^2\phi)\, d\phi,$$

the integral of which between the assigned limits

$$= a^2 \left(6 \cdot \frac{3}{4} \cdot \frac{1}{2} - 4 \cdot \frac{1}{2}\right) \frac{\pi}{2} = \frac{\pi}{8} a^2,$$

and the complete area inclosed by the roulette $=\frac{1}{2}\pi a^2$, i.e. it is half the area of the generating circle of the cycloid.

To find the length of the arc of the roulette, we must take ϕ from o to $\sin^{-1}\sqrt{\frac{2}{3}}$, and then from $\phi=\sin^{-1}\sqrt{\frac{2}{3}}$ to $\phi=\frac{\pi}{2}$, and add together the numerical values of the results. This will give one-fourth of the whole arc.

If ϕ be taken from o to $\frac{\pi}{2}$ we obtain $\sigma=\frac{\pi a}{2}$, so that the difference between the arcs OE and EC of the roulette is half the perimeter of the generating circle of the cycloid.

43. *A circle rolls on an equal circle, carrying a tangent; it is required to determine the nature of the roulette produced by the tangent.*

Let $OY=p$ (fig. 27), $A'OP=ACP=\theta$, and $YOA'=\phi$, OY being the perpendicular on the carried line.

Then $$p=2a\cos\theta-a, \text{ and } \phi=2\theta;$$

$$\therefore\ p=2a\cos\frac{\phi}{2}-a,$$

the tangential polar equation of the envelope.

Next $$\frac{d\sigma}{d\phi}=p+\frac{d^2p}{d\phi^2}=\frac{3a}{2}\cos\frac{\phi}{2}-a;$$

$$\therefore\ \sigma=3a\sin\frac{\phi}{2}-a\phi,$$

the intrinsic equation, measuring σ from A'.

Observing that the radius of curvature

$$=\frac{3a}{2}\cos\theta-a,$$

we see that a cusp occurs when $\cos\theta = \frac{2}{3}$. There are therefore two cusps, corresponding to the positive and negative values of θ.

When $\theta = \frac{\pi}{3}$, $p = 0$.

When $\cos\theta = \frac{1}{3}$, it will be seen by a figure that the tangent passes through the other end of the diameter through P, and that the envelope then crosses the circle at a point V distant $\frac{2a}{3}$ from P. It will also be found that the tangent at the cusp meets the diameter $A'O$ in the same point T at which it is intersected by the envelope itself.

Putting together all these considerations we obtain the figure (fig. 28), the curve being an involute of a two-cusped epicycloid.

The element of area swept over by PQ, i. e.

$$\begin{aligned} PQQ'P' &= r\cos\alpha\delta s - \frac{1}{2}r^2\delta\phi \\ &= r\delta\theta\,(a\cos\alpha - r) \\ &= r\delta\theta\,(2a\cos\theta - a) \\ &= pr\delta\theta. \end{aligned}$$

When $\theta > \frac{\pi}{3}$, this expression is still the same if we write for p its numerical value $a - 2a\cos\theta$, and any portion of the area is thus found by a simple integration.

GLISSETTES.

44. GLISSETTES are the curves traced out by points, or enveloped by curves, carried by a curve, which is made to slide between given points or given curves.

Thus if an ellipse slide between two fixed straight lines at right angles to each other, the glissette traced by its centre is the arc of a circle.

Again, if a straight line, of given length, slide between two fixed straight lines at right angles to each other, the glissette of any point in the line is an ellipse.

In this case, if p be the perpendicular from the intersection of the fixed lines on the sliding line (length a), and ϕ its inclination to one of the fixed lines,

$$p = \frac{a}{2} \sin 2\phi;$$

the envelope-glissette is therefore a four-cusped hypocycloid.

45. *A curve slides between two straight lines at right angles; to find the glissette of any carried point.*

Let the tangential polar equation of the curve, referred to the carried point, be

$$p = f(\phi);$$

then, if x, y be the perpendiculars from the carried point on the two fixed tangents,

$$x = f(\phi), \text{ and } y = f\left(\phi + \frac{\pi}{2}\right),$$

and the elimination of ϕ will give the rectangular equation the glissette.

If the two fixed lines be not at right angles, but inclin to each other at an angle $\pi - \alpha$, and if x, y be the obliqu co-ordinates of the carried point, we shall have to elimina ϕ between the equations

$$x \sin \alpha = f(\phi), \;\; y \sin \alpha = f(\phi + \alpha).$$

Ex. *An ellipse slides between two straight lines incline to each other at an angle $\pi - \alpha$; to find the path of th centre.*

In this case, we can write the equations in the form

$$x^2 \sin^2 \alpha = a^2 \cos^2 \left(\phi - \frac{\alpha}{2}\right) + b^2 \sin^2 \left(\phi - \frac{\alpha}{2}\right),$$

$$y^2 \sin^2 \alpha = a^2 \cos^2 \left(\phi + \frac{\alpha}{2}\right) + b^2 \sin^2 \left(\phi + \frac{\alpha}{2}\right),$$

and the result of the elimination is

$$\cos^2 \alpha \, (a^2 + b^2)^2 \, (x^2 - y^2)^2 + \sin^2 \alpha \, (a^2 - b^2)^2 \, (x^2 + y^2)^2$$
$$- 2 \, (a^2 + b^2) \, (a^2 - b^2)^2 \, (x^2 + y^2) + (a^4 - b^4)^2 = 0.$$

46. *The following theorem is of great importance.*

Any state of motion of a plane area in its own plane can be represented by a state of rotation about a point.

Any plane area is fixed if two of its points are fixed; and, if the motions of two of its points are given, the motion of the area is given.

We must observe, however, that we cannot assign an arbitrary motion to two points; the restriction existing that the velocities of the two points in the direction of the line joining them must be the same.

Suppose that two points P and Q are in motion in the directions PT and QV, and that PE, QE are drawn perpendicular to those lines respectively, and meeting in E.

It is clear that the motion of P may be represented by a state of rotation about any point in the line PE, and that of Q by a rotation about any point in QE.

Hence, both motions are represented by a state of rotation about E.

This point E is called the *instantaneous centre of rotation*, and the motion of any point R in the area is, at the instant considered, at right angles to RE.

Thus, if a curve slide between two given lines, the intersection of the normals at the points of contact is the instantaneous centre.

47. *Any motion of an area in its own plane can be represented by the rolling of a certain determinate curve on another determinate curve.*

Let $PQRS$ (fig. 29) be the curve traced out in space by the successive positions of the instantaneous centre, and let the angular velocities of the moving area corresponding to each position of the instantaneous centre be known.

Then, if P, Q, R, ... be successive positions of the centre at given infinitesimal intervals of time, the lines in the moving area QP, RQP', $SRq'p'$, ... will turn successively into the positions QP', $Rq'p'$, $Srqp$, &c.

Hence the motion can be represented by rolling the curve $pqrs$ ... on the curve $PQRS$....

But the curve $pqrs$ ... is the locus on the moving area of the instantaneous centre, and the two curves are therefore determinate.

48. Ex. 1. *A straight line* AB *slides between two fixed lines at right angles to each other.*

In this case the locus of the instantaneous centre, E, with regard to the fixed lines Ox, Oy, is a circle, centre O and radius OE (fig. 30), while the locus of E with regard to AB is a circle on AB as diameter, that is a circle.

Hence, the carried locus of E is a circle, the radius of which is half that of the circle which is the fixed locus of E.

But, if a circle roll internally on a fixed circle of double its size, any carried point traces an ellipse*.

Hence, the path of any point Q, connected with AB, and carried with it, is an ellipse.

It will also be found by an application of the formula of Art. (34), that the envelope of any straight line carried by the circle is an involute of a four-cusped hypocycloid.

Hence, the glissette produced by any straight line carried by AB is an involute of a four-cusped hypocycloid.

49. Ex. 2. *Two straight lines,* AB, AC, *containing a given angle, move so that* AB, AC *pass respectively through fixed points* P *and* Q.

The point E being the instantaneous centre, AE is the diameter of the circle about PAQ, and is constant (fig. 31).

Hence, the carried locus of E is a circle, centre A and radius AE, while the fixed locus is the circle PAQ.

The sliding motion is therefore equivalent to the motion produced by rolling a circle, with internal contact, on a circle of half its radius.

The glissette for a carried point is therefore a hypotrochoid, and the glissette for a carried straight line is a circle.

This last can be shewn by the formula of Art. (34), for if EN be the carried line, at a distance CN from C, $OPQ = \alpha$, $\rho' = a$, and $\rho = -2a$ (fig. 32);

---

* See Appendix to *Geometrical Conics.*

$$\therefore \frac{d\sigma}{d\phi} = PQ + 2a\cos OCA'$$
$$= CN,$$

i.e. the curvature is constant; or, by direct geometry, we can see that if A roll from A', and G be the other end of the diameter $A'OG$,

$$GQ = CN,$$

and therefore the envelope of EN is a circle, centre G and radius CN.

50. Ex. 3. *An involute of a circle slides between two straight lines at right angles to each other.*

Let Ox, Oy be the fixed lines (fig. 33),

$$OP = x, \text{ and } OQ = y,$$

co-ordinates of E the instantaneous centre.

Then, $$x = \left(\frac{\pi}{2} + \phi\right) a - a,$$

and $$y = a\phi + a;$$

$$\therefore y - x = \left(2 - \frac{\pi}{2}\right) a.$$

Hence, the fixed locus of E is a straight line, and the path of E relative to the curve AP is clearly a circle, centre C.

Hence the point glissettes are cycloids and trochoids, and the envelopes of carried straight lines are involutes of a cycloid.

51. *A given curve slides between two straight lines at right angles to each other; to find the locus, with regard to the lines, of the instantaneous centre.*

Let x, y be co-ordinates of E, C the centre of curvature at P, and CK an element of the evolute at P (fig. 34).

Then, by turning the curve round E through a small angle $\delta\phi$, and taking K such that the angle of deflection of the arc CK is $\delta\phi$, the tangent at K will glide into the position $K'P'$, and we have

$$\delta x = - PP' = - (\rho - y)\,\delta\phi;$$

$$\therefore \frac{dx}{d\phi} - y = - \rho.$$

Similarly, we shall obtain

$$\frac{dy}{d\phi} + x = \rho',$$

if ρ' be the radius of curvature at Q.

Hence, if the intrinsic equation be known,

$$\rho = f(\phi) \text{ and } \rho' = f\left(\phi + \frac{\pi}{2}\right),$$

and the two equations we have obtained will give x and y in terms of ϕ, and reduce the solution of the problem to the elimination of ϕ.

Take, for instance, the case of Art. (50), in which

$$\rho = a\phi,$$

and therefore

$$\frac{dx}{d\phi} - y = - a\phi,$$

$$\frac{dy}{d\phi} + x = a\left(\frac{\pi}{2} + \phi\right).$$

Integrating these equations, and remembering that when

$$\phi = 0,\ x = \frac{\pi a}{2} - a, \text{ and } y = a,$$

we shall obtain the equations of Art. (50).

52. The preceding equations determine the tangent and normal at E, and, by differentiation, the curvature of the locus of E.

A geometrical construction may however be given, for since $EN = PP' = EC\delta\phi$, and $E'N = EC'\delta\phi$ (fig. 35);

$$\therefore EN : E'N :: EC : EC',$$

and the angle

$$NEE' = ECC'.$$

Hence, if EF be perpendicular to EE',

the angle $$FEC' = \frac{\pi}{2} - E'EN = EC'F,$$

and F is the middle point of CC'.

If then D, D' be the centres of curvature consecutive to C, C' in their new positions, and F' the middle point of DD', the lines EF, $E'F'$ intersect in the centre of curvature at E.

For the carried locus, i.e. the locus of E with regard to the sliding curve, we observe that the tangent and normal are the same as for the locus above considered.

For the curvature, however, let CK, $C'K'$ (fig. 36), be elements of the evolute having the same deflection $\delta\phi$, E' the intersection of the tangents at K and K', and F' the middle point of KK'; then G, the intersection of EF and $E'F'$, is the centre of curvature.

In either case, CK being an infinitesimal of the first order,

$$\text{arc } EE' = EF.\,2\delta\phi = CC'\delta\phi,$$

or ultimately $$\frac{d\sigma}{d\phi} = CC'.$$

Also, if the FEC' in (figure 35) $= \psi$,

$$\tan\psi = \frac{EC}{EC'},$$

and, if EC and EC', i.e. $\rho - y$ and $\rho' - x$, can be found in terms of ϕ, we shall have sufficient data for the determination of $\frac{d\sigma}{d\psi}$, the radius of curvature.

For the carried locus (fig. 36), we must add to ψ the inclination of EC' to some fixed line in the moving area.

53. Taking the general case of any motion of a plane area (which however is reducible to a case of roulettes), and supposing that the two loci of E can be found, or, which is the same thing, the fixed locus of E, and the rates of rotation as compared with the increase of arc along the locus

of E, we can find simple expressions for the curvatures of curves traced by carried points, or enveloped by carried lines.

To find the curvature of the curve traced by a carried point Q.

Let EE' ($\delta\sigma$) be an element of the fixed locus of E (fig. 37), and α the inclination of EQ (r) to the normal at E.

Then $$VQ : EQ :: QQ' : QQ' - E'N,$$

if $E'N$ be perpendicular to EQ;

or radius of curvature $= VQ$

$$= \frac{r^2\delta\phi}{r\delta\phi - \delta\sigma \,.\, \cos\alpha}.$$

If we take the case of Art. (21),

$$\delta\phi = \delta\sigma\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

and we obtain the formula of that article.

If we take the case of Art. (52),

$$\delta\sigma = CC'\delta\phi,$$

therefore, radius of curvature

$$= \frac{r^2}{r - \cos\alpha \,.\, CC'}.$$

54. *To find the curvature of the envelope of a carried straight line.*

Let $r = EQ$ be the perpendicular from E on the line (fig. 38).

Then $$\delta s = Qq + qQ'$$
$$= \delta\sigma\cos\alpha + EQ\delta\phi,$$

and $$\frac{ds}{d\phi} = r + \cos\alpha\frac{d\sigma}{d\phi}.$$

For instance, in the case of Art. (52), radius of curvature

$$= r + \cos\alpha \,.\, CC'.$$

55. *To find the curvature of the envelope of any carried curve.*

Let EQ, $E'q$ be normals to the carried curve, and ρ'' the radius of curvature at Q (fig. 39).

Then
$$\begin{aligned}\delta s &= Qq + Q'q\\ &= \delta\sigma\cos\alpha\,\frac{\rho''}{\rho''+EQ} + EQ\delta\phi,\end{aligned}$$

and if
$$QVQ' = \delta\psi,$$
$$\delta\phi - \delta\psi = E'OE = \frac{\delta\sigma\cos\alpha}{\rho''+r};$$

therefore, radius of curvature of envelope

$$\begin{aligned}&= \frac{ds}{d\psi}\\ &= \frac{\dfrac{\rho''\cos\alpha}{\rho''+r}\,d\sigma + r\,d\phi}{d\phi - \dfrac{\cos\alpha\,d\sigma}{\rho''+r}}.\end{aligned}$$

For instance, taking the case of Art. (37), we have

$$d\phi = d\sigma\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

and we fall upon the formula of that article.

56. *A triangle moves in a plane so that two of its sides slide on fixed curves; it is required to find the envelope of the third side.*

Let α, β, γ be the perpendiculars from any fixed point on the sides a, b, c; then

$$a\alpha + b\beta + c\gamma = 2 \text{ (area of triangle)} = 2\Delta.$$

Hence, if ϕ be the angle which one of the perpendiculars makes with any fixed line, and remembering that

$$\alpha + \frac{d^2\alpha}{d\phi^2}$$

is an expression for the radius of curvature of the envelope of the side BC, we obtain

$$a\rho_1 + b\rho_2 + c\rho_3 = 2\Delta,$$

where ρ_1, ρ_2, ρ_3 are the radii of curvature of the envelopes of the several sides.

If then two of these be given, the third is determined.

57. Ex. 1. *A given triangle moves so that two of its sides touch fixed circles.*

In this case ρ_1 and ρ_2 are constant; ρ_3 is therefore constant, and the third side always touches a fixed circle.

This includes the example of Art. (49) as a particular case.

A direct geometrical proof may be also given.

Firstly, let the sides AB, AC pass through fixed points P, Q.

Through A draw AF parallel to BC, and meeting in F the fixed circle which is the locus of A (fig. 40).

The angle $FAP = ABC$, and therefore the arc FP is constant, and F is a fixed point.

Also, the perpendicular FG from F on BC is equal to the altitude of the triangle ABC; therefore BC always touches a circle, the centre of which is at F.

Secondly, let AB, AC touch fixed circles having their centres at P and Q.

Through P and Q draw lines PA', QA' parallel to AB, AC, and meeting BC in B' and C': then, as before, $B'C'$ touches a fixed circle.

Ex. 2. *An isosceles right-angled triangle moves so that its equal sides slide on the arc of a cycloid.*

In this case, $a = b$,

$$\rho_1 = d \sin\phi, \quad \rho_2 = d \cos\phi,$$

and therefore $c\rho_3 = 2\Delta - ad\sqrt{2}\sin\left(\phi + \frac{\pi}{4}\right)$;
hence it appears that the hypothenuse envelopes an involute of a cycloid.

58. *Two straight lines* AB, AC, *inclined to each other at a given angle* λ, *and carrying a line* AD, *slide on fixed curves; it is required to find the envelope of* AD.

This is a particular case of Art. (56), and if μ, ν be the inclinations of *AD* to *AB* and *AC*, the first equation of Art. (56) becomes

$$\alpha\sin\mu + \beta\sin\nu + \gamma\sin\lambda = 0.$$

Hence, $\rho_1\sin\mu + \rho_2\sin\nu + \rho_3\sin\lambda = 0$,
and if ρ_1 and ρ_2 are given, ρ_3 is determined.

59. Ex. 1. Thus, from Ex. (2), Art. (57), we obtain the following result.

If two straight lines at right angles to each other slide on the arc of a cycloid, the straight line bisecting the angle between them always touches a cycloid.

Ex. 2. AB, AC *slide completely round an oval curve.*

In this case, if

$$-\rho_3 = \frac{d\sigma}{d\phi},$$

$$\sin\lambda\frac{d\sigma}{d\phi} = \sin\mu\frac{ds}{d\phi} + \sin\nu\frac{ds'}{d\phi},$$

ds, *ds'* being elements at the points of contact of *AB*, *AC*.

Hence, if l be the perimeter of the oval, and σ the whole arc enveloped by *AD*,

$$\sin\lambda\,.\,\sigma = l(\sin\mu + \sin\nu),$$

$$\text{or}\quad \sigma = l\frac{\sin\frac{\mu+\nu}{2}}{\sin\frac{\mu-\nu}{2}}.$$

We have put

$$\frac{d\sigma}{d\phi} = -\rho_3,$$

because if α and β are positive, γ is necessarily negative.

EXAMPLES.

1. If a circle roll inside a circle of twice its radius, the path of a point on its circumference is a straight line, and the path of any point, not on the circumference, is an ellipse.

2. The roulette, on a straight line, of the pole of an epicycloid is an ellipse.

3. Shew that the intrinsic equation of a cardioid is of the form

$$s = c\left(1 - \cos\frac{\phi}{3}\right),$$

and that it is the epicycloid due to the rolling of a circle, with internal contact, on a fixed circle of half its diameter.

4. A parabola slides on two straight lines at right angles to each other; prove that its vertex and focus describe respectively the curves

$$x^2y^2(x^2 + y^2 + 3a^2) = a^6, \text{ and } x^2y^2 = a^2(x^2 + y^2).$$

5. The roulette, on a circle, of the pole of an equiangular spiral is an involute of another circle.

6. The envelope of the pedal line of a triangle is a three-cusped hypocycloid, having its centre at the centre of the nine point circle.

Note. The pedal line of a triangle is the straight line joining the feet of the perpendiculars let fall upon the sides from any point of the circumscribing circle.

7. A parabola rolls symmetrically on an equal parabola; find the path of the focus, and prove that the path of the vertex is the cissoid

$$y^2(2a-x)=x^3.$$

8. An involute of a circle rolls on a straight line; the roulette of the centre of the circle is a parabola.

9. The ellipse $\frac{c}{r}=1+e\cos\theta$ rolls on a straight line; the path of the focus is given by the equation

$$e^2=1+\frac{2c}{y}\frac{dx}{ds}+\frac{c^2}{y^2};$$

and the path of the centre by

$$y^4\left(\frac{ds}{dx}\right)^2=(a^2+b^2)\,y^2-a^2b^2,$$

$2a$ and $2b$ being the axes.

10. The roulette, on a straight line, of the centre of a rectangular hyperbola is

$$\frac{dx}{dy}=\frac{y^2}{\sqrt{a^4-y^4}}.$$

11. A helix rolls on a straight line which it always touches, while its axis moves in a plane; any point of the helix traces out a cycloid.

12. The roulette, on a straight line, of the pole of an hyperbolic spiral, $r\theta=c$, is

$$\frac{dy}{dx}=\frac{y}{\sqrt{c^2-y^2}};$$

and of the pole of the curve $c^n p=r^{n+1}$, is

$$\left(\frac{ds}{dx}\right)^{n+1}=\left(\frac{c}{y}\right)^n.$$

13. A cycloid rolls on a straight line; the locus of its vertex is given by the equations

$$x = 2a(\sin\phi - \phi\cos\phi), \quad y = 2a\phi\sin\phi,$$

the origin being the point of the line over which the vertex passes.

14. A curve rolls symmetrically on an equal curve, carrying an involute; the envelope of this involute is an involute of the fixed curve.

15. If a curve roll on a straight line, the curvature of a point roulette varies as $\frac{d}{dp}\left(\frac{p}{r}\right)$, p and r being referred to the point.

If the curve be

$$\frac{a}{r} = 1 + \sec\alpha\sin(\theta\sin\alpha),$$

the roulette is a circle.

16. The curve $P'Q$ rolls on the curve PQ, P' passing over P; the roulette of P' is, in the neighbourhood of P, a semi-cubical parabola, of which the parameter is

$$\frac{9\rho\rho'(\rho+\rho')}{2(\rho+2\rho')^2},$$

ρ and ρ' being the radii of curvature at the point of contact.

17. If a given arc of a curve roll, first externally, and then internally, over the same arc of a fixed curve, the sum of the arcs of the roulettes of the same point is independent of the nature of the fixed curve.

The same independence also exists for the sum of the areas swept over by the line joining the carried point with the point of contact.

18. If a parabola roll on a straight line, the envelope of its directrix is a catenary.

19. A catenary, $s = c \tan \phi$, rolls symmetrically on an equal catenary; the intrinsic equation to the envelope of its axis is

$$\frac{d\sigma}{d\psi} = c \log \tan \frac{\pi - \psi}{4} + \frac{c}{2} \tan \frac{\psi}{2} \sec \frac{\psi}{2}.$$

20. The portion of the tangent at any point of a three-cusped hypocycloid, contained within the curve, is of constant length.

21. If an oval curve roll on a straight line, prove that the area traced out by any point O in the curve will exceed the area of the curve by $\frac{1}{2}\int_0^{2\pi} r^2 d\phi$, where r is the distance from O of any point P of the curve, and ϕ the angle which the tangent at P makes with some fixed line in the curve: apply this to find the area of a cycloid.

22. If an oval curve A roll upon an equal and similar curve B, so that the point of contact is a centre of similitude for each, then the whole area traced out by any point O when A has made a complete revolution, is twice the area which would have been traced out if the curve A had rolled upon a straight line.

23. Test the formula of Art. (27) by applying it to an ellipse, measuring r from the focus.

24. A parabola rolls on a straight line from one end of the latus rectum to the other; the length of the arc enveloped by the axis is

$$2a \log (2\epsilon).$$

25. A parabola rolls symmetrically on an equal parabola, from one end of the latus rectum ($4a$) to the other; the length of arc enveloped by the axis is

$$2a \log (4\epsilon).$$

26. A diameter of a circle rolls on a curve; the envelope of the carried circle consists of two involutes of the curve.

27. A circle, radius b, rolls on a fixed circle of radius a; the area between the fixed circle, and the envelope of a diameter for a half roll from one end of the diameter to the other is equal to

$$\frac{\pi b^2}{4a}(3a+b).$$

28. The lemniscate $r^2 = a^2 \cos 2\theta$ rolls on a straight line; the tangential polar equation of the roulette produced by its pole is

$$\frac{dp}{d\phi} + p \tan\phi = a \tan\phi \sqrt{\sin\phi};$$

and the intrinsic equation to the envelope of its axis is

$$\frac{ds}{d\phi} = \frac{a}{6\sqrt{\cos\frac{2\phi}{3}}}\left(5 \sin\phi - 3 \sin\frac{\phi}{3}\right).$$

29. A circle rolls on a fixed circle; the envelope of any carried straight line is an involute of an epicycloid.

30. A catenary rolls on a straight line; the envelope of any carried straight line is an involute of a parabola.

31. An ellipse rolls, symmetrically, on an equal ellipse; prove that the whole length of the arc enveloped by its axis is

$$2a\left(1 + \frac{1-e^2}{e}\log\frac{1+e}{1-e}\right).$$

32. A curve, carrying a point, rolls on a straight line, and then, symmetrically, on an equal curve; prove that after rolling over the same arc in each case, the radii of curvature of the roulettes, and the distance of the point from the point of contact, are in Harmonic Progression.

33. In the same case, if a straight line be carried, the radii of curvature of the roulettes, and the distance of the line from the point of contact, are in Arithmetic Progression.

34. An ellipse rolls on a straight line; the length of the envelope of its axis between two consecutive cusps is

$$a\left(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right).$$

35. A circle rolls on a straight line; shew that the envelope of any carried straight line is an involute of a cycloid; and trace the figures corresponding to the cases in which the distance of the carried line from the centre is greater than, equal to, or less than the diameter of the circle.

36. An ellipse rolls on a straight line, find the intrinsic equation to the envelope of its directrix, and shew that it has two cusps if the eccentricity is greater than $\frac{\sqrt{5}-1}{2}$, and that, if $e<\frac{\sqrt{5}-1}{2}$, the length of the arc of the roulette, corresponding to a complete roll of the ellipse, is $\frac{4\pi a}{e}$.

37. A straight line rolls on a curve, $s=f(\phi)$, carrying a straight line inclined to it at an angle α; the envelope roulette is

$$s=f(\phi)+\cos\alpha f'(\phi).$$

If the curve be an epicycloid, or hypocycloid, the envelope roulette is of the same class.

38. A straight line slides on a curve having always the same point in contact; the motion is identical with that of rolling a perpendicular straight line on the evolute.

39. Hence, shew that the envelope of the lines drawn through each point of an epicycloid at a constant angle to the tangent is also an epicycloid.

40. If a parabola, latus rectum $4a$, slide between two straight lines at right angles to each other the glissettes pro-

duced are the same as the roulettes produced by a parabola, latus rectum a, rolling on the curve

$$a^2 (x^2 + y^2)^3 = x^4 y^4.$$

41. An ellipse slides on a straight line, always touching it at the same point; the path of its centre is the curve

$$x^2 y^2 = (a^2 - y^2)(y^2 - b^2).$$

42. A smooth rigid curve bent into a curve turns round a fixed point in its own plane, and pushes a particle before it in a straight line; find the form of the curve.

43. A straight line APB carrying a point P slides with its ends A and B on the arc of a closed curve. If $PA = a$ and $PB = b$, the difference between the area of the curve and of the locus of P is equal to πab.

44. An involute of a circle slides on a straight line, always touching it at the same point; the glissettes of a point and a straight line are respectively a trochoid and an involute of a cycloid.

45. A parabola slides on a straight line touching it at a fixed point P.

If the normal at P meet the axis in G and GR be drawn parallel to SP and equal to one fourth of the latus rectum, the normal to the path of the focus is parallel to PR.

Shew also that the path of the focus is an hyperbola.

46. The angle BAC slides over two fixed circles; prove that the point glissettes are the same as the roulettes of an ellipse on a circle.

THE END.

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